The Riemann Zeta Function Across Number Bases: A Study of Notational Elegance

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January 2, 2026

Abstract

We explore the representation of the Riemann zeta function and its special values in various number bases, with particular emphasis on base 4 and base 12. We demonstrate that while the mathematical properties of the zeta function remain invariant under change of base, certain bases provide notational elegance for specific values. Notably, base 12 offers a remarkably simple representation of the Ramanujan summation $\zeta(-1) = -\frac{1}{12}$, and we explore connections to the Riemann Hypothesis through the critical line $\text{Re}(s) = \frac{1}{2}$.

1 Introduction

The Riemann zeta function is one of the most important functions in mathematics, defined for $\text{Re}(s) > 1$ by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Through analytic continuation, this function extends to the entire complex plane (except for a simple pole at $s = 1$), yielding remarkable values such as:

$$\zeta(-1) = -\frac{1}{12} \tag{2}$$

While the mathematical properties of $\zeta(s)$ are independent of our choice of number base, the representation of its values can take on striking simplicity in certain bases. This paper explores how different bases illuminate various aspects of the zeta function.

2 The Zeta Function: Base-Independent Mathematics

2.1 Fundamental Definition

The value of the zeta function at any point s is a mathematical constant, independent of how we choose to represent numbers. Whether we write:

$$\zeta(2) = \frac{\pi^2}{6} \quad \text{(base 10)} \tag{3}$$

$$\zeta(2) = \frac{\pi^2}{6_{12}} \quad \text{(base 12)} \tag{4}$$

$$\zeta(2) = \frac{\pi^2}{12_4} \quad \text{(base 4)} \tag{5}$$

we are referring to the same mathematical reality—only the notation changes.

2.2 The Riemann Hypothesis

The Riemann Hypothesis, one of the most important unsolved problems in mathematics, states that all non-trivial zeros of $\zeta(s)$ lie on the critical line:

$$\text{Re}(s) = \frac{1}{2} \tag{6}$$

This relationship between $\frac{1}{2}$ (the critical line) and $-\frac{1}{12}$ (the value at $s = -1$) provides a deep connection in the theory of the zeta function.

3 The Zeta Function in Base 10

In our familiar decimal system, the zeta function is written:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \tag{7}$$

Key values include:

$$\zeta(2) = \frac{\pi^2}{6} = 1.644934\ldots \tag{8}$$

$$\zeta(0) = -\frac{1}{2} = -0.5 \tag{9}$$

$$\zeta(-1) = -\frac{1}{12} \approx -0.08333\ldots \tag{10}$$

The Riemann Hypothesis states that non-trivial zeros satisfy:

$$\text{Re}(s) = 0.5 \tag{11}$$

4 The Zeta Function in Base 4

4.1 Notation in Base 4

In base 4 (quaternary), we use digits 0, 1, 2, 3. The place values are powers of 4:

$$1_4 = 1_{10} \tag{12}$$
$$2_4 = 2_{10} \tag{13}$$
$$3_4 = 3_{10} \tag{14}$$
$$10_4 = 4_{10} \tag{15}$$
$$11_4 = 5_{10} \tag{16}$$
$$12_4 = 6_{10} \tag{17}$$
$$13_4 = 7_{10} \tag{18}$$
$$20_4 = 8_{10} \tag{19}$$

4.2 The Zeta Function in Base 4

$$\zeta(s) = \sum_{n=1_4}^{\infty} \frac{1}{n^s} = \frac{1}{1_4^s} + \frac{1}{2_4^s} + \frac{1}{3_4^s} + \frac{1}{10_4^s} + \frac{1}{11_4^s} + \frac{1}{12_4^s} + \cdots \tag{20}$$

4.3 Key Values in Base 4

Since $12_{10} = 30_4$ (because $12 = 3 \times 4 + 0$), we have:

$$\zeta(-1) = -\frac{1}{12_{10}} = -\frac{1}{30_4} \tag{21}$$

The Basel problem becomes:

$$\zeta(2) = \frac{\pi^2}{12_4} \tag{22}$$

The critical line of the Riemann Hypothesis:

$$\text{Re}(s) = \frac{1}{2_{10}} = 0.2_4 \tag{23}$$

This is because $\frac{1}{2}$ in base 4 is 0.2_4 (since $2_4 \times 0.2_4 = 1_4$).

5 The Zeta Function in Base 12

5.1 Notation in Base 12

Base 12 (duodecimal or dozenal) uses digits 0-9 and two additional symbols. We adopt the common convention:

- A represents ten ($\text{A}_{12} = 10_{10}$)

- B represents eleven ($\text{B}_{12} = 11_{10}$)

The place values are powers of 12:

$$1_{12} = 1_{10} \tag{24}$$
$$2_{12} = 2_{10} \tag{25}$$
$$\vdots \tag{26}$$
$$9_{12} = 9_{10} \tag{27}$$
$$\text{A}_{12} = 10_{10} \tag{28}$$
$$\text{B}_{12} = 11_{10} \tag{29}$$
$$10_{12} = 12_{10} \tag{30}$$
$$11_{12} = 13_{10} \tag{31}$$
$$12_{12} = 14_{10} \tag{32}$$

5.2 The Zeta Function in Base 12

$$\zeta(s) = \sum_{n=1_{12}}^{\infty} \frac{1}{n^s} = \frac{1}{1_{12}^s} + \frac{1}{2_{12}^s} + \cdots + \frac{1}{\text{B}_{12}^s} + \frac{1}{10_{12}^s} + \frac{1}{11_{12}^s} + \frac{1}{12_{12}^s} + \cdots \tag{33}$$

5.3 The Elegant Representation of $\zeta(-1)$

Observation 1. *In base 12, the value $\zeta(-1) = -\frac{1}{12}$ has a remarkably simple representation:*

$$\zeta(-1) = -\frac{1}{10_{12}} \tag{34}$$

This is the dozenal analog of writing $\frac{1}{10}$ in base 10. Just as $0.1_{10} = \frac{1}{10}$ in decimal, we have $0.1_{12} = \frac{1}{12}$ in dozenal.

5.4 The Ramanujan Summation in Base 12

The famous regularized sum can now be written:

$$1_{12} + 2_{12} + 3_{12} + 4_{12} + \cdots = -\frac{1}{10_{12}} = -0.1_{12} \tag{35}$$

This representation is particularly elegant—the sum of all positive integers equals negative one-tenth (in base 12).

5.5 Other Key Values in Base 12

The Basel problem:

$$\zeta(2) = \frac{\pi^2}{6_{12}} \tag{36}$$

The critical line of the Riemann Hypothesis:

$$\text{Re}(s) = \frac{1}{2}_{10} = 0.6_{12} \tag{37}$$

This is because $\frac{1}{2}$ in base 12 is 0.6_{12} (since $2_{12} \times 0.6_{12} = 1_{12}$, or equivalently $\frac{12}{2} = 6$).

6 Comparative Analysis

6.1 Notational Simplicity

Value	Base 10	Base 4	Base 12
$\zeta(-1)$	$-\frac{1}{12}$	$-\frac{1}{30_4}$	$-\frac{1}{10_{12}}$
$\zeta(0)$	$-\frac{1}{2}$	$-\frac{1}{2_4}$	$-\frac{1}{2_{12}}$
$\zeta(2)$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{12_4}$	$\frac{\pi^2}{6_{12}}$
Critical line	0.5	0.2_4	0.6_{12}

Table 1: Representation of key zeta function values in different bases

6.2 Why Base 12 is Natural for $\frac{1}{12}$

Base 12 provides exceptional notational elegance for $\zeta(-1)$ because:

1. The denominator 12 becomes the base unit "10" in base 12
2. The expression $-\frac{1}{10_{12}}$ parallels the familiar $\frac{1}{10}$ in base 10
3. Fractions with denominator 12 have simple representations: $\frac{1}{12} = 0.1_{12}$, $\frac{1}{6} = 0.2_{12}$, $\frac{1}{4} = 0.3_{12}$, $\frac{1}{3} = 0.4_{12}$, etc.

6.3 Connection to the Riemann Hypothesis

The relationship between $\zeta(-1) = -\frac{1}{12}$ and the critical line $\text{Re}(s) = \frac{1}{2}$ takes on different notational forms:

Base	$\zeta(-1)$	Critical Line
Base 10	$-\frac{1}{12}$	$\text{Re}(s) = 0.5$
Base 4	$-\frac{1}{30_4}$	$\text{Re}(s) = 0.2_4$
Base 12	$-\frac{1}{10_{12}}$	$\text{Re}(s) = 0.6_{12}$

Table 2: The special values and the critical line in different bases

7 Philosophical Considerations

7.1 The Invariance of Mathematical Truth

This exploration demonstrates a fundamental principle: *mathematical truth is independent of representation*. The Riemann zeta function has the same zeros, the same special values, and the same deep properties regardless of which base we use to express numbers.

7.2 Notational Elegance and Insight

However, choice of base can provide:

- **Aesthetic appeal**: Base 12 makes $\zeta(-1) = -\frac{1}{10_{12}}$ visually cleaner
- **Computational convenience**: Certain bases may simplify numerical calculations
- **Conceptual clarity**: The right notation can make patterns more apparent
- **Historical perspective**: Our attachment to base 10 is cultural, not mathematical

7.3 The Dozenal Movement

Advocates of base 12 (the Dozenal Society) argue that 12 is superior to 10 because:

- 12 has more divisors: 1, 2, 3, 4, 6, 12 (versus 1, 2, 5, 10 for base 10)
- Many common fractions have finite representations: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$
- The value $\zeta(-1) = -\frac{1}{10_{12}}$ is just one example of notational elegance

8 Extensions and Further Questions

8.1 Other Bases

We might ask: in what base does $\zeta(-1)$ take on other interesting forms?

- **Base 24**: $\zeta(-1) = -\frac{1}{12_{10}} = -\frac{1}{\mathrm{C}_{24}}$ (using C for 12 in base 24)
- **Base 6**: $12_{10} = 20_6$, so $\zeta(-1) = -\frac{1}{20_6}$
- **Base 144**: $\zeta(-1) = -\frac{1}{10_{144}}$ (since $144 = 12^2$)

8.2 Connection to Other Constants

Could other mathematical constants have particularly elegant representations in specific bases?

- e in base e (though not an integer base)
- π in various bases
- The golden ratio ϕ in base ϕ

8.3 Zeta Function at Other Points

What about other special values?

$$\zeta(0) = -\frac{1}{2} = -\frac{1}{2_b} \text{ (same in all bases)} \tag{38}$$

$$\zeta(-3) = \frac{1}{120_{10}} = \frac{1}{500_4} = \frac{1}{\text{A}0_{12}} \tag{39}$$

9 Conclusion

The Riemann zeta function, a cornerstone of analytic number theory, reveals different aspects of its beauty depending on the base in which we express its values. While the mathematical content remains invariant, base 12 offers a particularly elegant representation of the Ramanujan summation:

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{10_{12}} = -0.1_{12} \tag{40}$$

This simplicity mirrors the familiar 0.1 notation in decimal and highlights the arbitrary nature of our cultural preference for base 10.

The critical line of the Riemann Hypothesis, $\text{Re}(s) = \frac{1}{2}$, connects intimately with this value. Whether we write it as 0.5_{10}, 0.2_4, or 0.6_{12}, the profound mysteries surrounding the distribution of prime numbers and the location of the zeta zeros remain—a reminder that deep mathematical truth transcends notation.

Future research might explore:

- Computational advantages of different bases for numerical evaluation of $\zeta(s)$
- Whether certain bases make patterns in the zeros more apparent
- Educational applications: does learning about $\zeta(-1)$ in base 12 provide better intuition?

In the end, this exercise demonstrates that mathematics is not about the symbols we use, but about the eternal relationships they represent. The number $-\frac{1}{12}$, whether written as $-0.08\overline{3}_{10}$, -0.02_4, or -0.1_{12}, remains a fixed point in the mathematical universe—a constant beacon in the beautiful and mysterious landscape of the Riemann zeta function.

References

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