Analytic Continuation of Divergent Series: A Brief Exploration

Discussion Summary

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Abstract

We explore several famous infinite series equivalences and investigate the analytic continuation of divergent series through the Riemann zeta function. In particular, we examine the harmonic series and the series of odd and even positive integers, demonstrating how regularization techniques assign finite values to these otherwise divergent sums.

1 Famous Infinite Series Equivalences

We begin by recalling some of the most celebrated infinite series in mathematics.

1.1 Geometric Series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for } |r| < 1 \tag{1}$$

1.2 Basel Problem

Solved by Euler in 1734:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{2}$$

1.3 Leibniz Formula for π

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{3}$$

1.4 Exponential Function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{4}$$

1.5 Natural Logarithm

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad \text{for } |x| \leq 1 \tag{5}$$

1.6 Trigonometric Functions

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{6}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{7}$$

2 The Harmonic Series

2.1 Divergence of the Harmonic Series

The harmonic series is defined as:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{8}$$

Theorem 1. *The harmonic series diverges to infinity.*

Proof. We group the terms as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \tag{9}$$

Observe that:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{10}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \tag{11}$$

Therefore, the series is greater than $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$, which clearly diverges. □

2.2 Analytic Continuation via the Riemann Zeta Function

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{12}$$

The harmonic series corresponds to $\zeta(1)$, which is a simple pole of the zeta function. Near $s = 1$, the zeta function has the Laurent expansion:

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \tag{13}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Unlike the Ramanujan summation $1+2+3+\cdots = -\frac{1}{12}$ (which comes from $\zeta(-1) = -\frac{1}{12}$), the harmonic series does not have a finite analytic continuation value.

2.3 The Alternating Harmonic Series

In contrast, the alternating harmonic series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2) \tag{14}$$

3 Analytic Continuation of Arithmetic Progressions

3.1 Sum of All Positive Integers

The famous Ramanujan summation gives:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \zeta(-1) = -\frac{1}{12} \tag{15}$$

3.2 Sum of Even Positive Integers

Proposition 1. *The regularized value of the sum of even positive integers is:*

$$2 + 4 + 6 + 8 + \cdots = -\frac{1}{6} \tag{16}$$

Proof. We have:

$$\sum_{n=1}^{\infty} 2n = 2\sum_{n=1}^{\infty} n = 2 \cdot \zeta(-1) = 2 \cdot \left(-\frac{1}{12}\right) = -\frac{1}{6} \tag{17}$$

□

3.3 Sum of Odd Positive Integers

Proposition 2. *The regularized value of the sum of odd positive integers is:*

$$1 + 3 + 5 + 7 + \cdots = \frac{1}{12} \tag{18}$$

Proof. Since every positive integer is either odd or even, we have:

$$\sum_{n=1}^{\infty} n = \sum_{\text{odd}} n + \sum_{\text{even}} n \tag{19}$$

Therefore:

$$\sum_{\text{odd}} n = \zeta(-1) - \sum_{\text{even}} n = -\frac{1}{12} - \left(-\frac{1}{6}\right) = -\frac{1}{12} + \frac{2}{12} = \frac{1}{12} \tag{20}$$

□

3.4 Verification

We verify our results:

$$\frac{1}{12} + \left(-\frac{1}{6}\right) = \frac{1}{12} - \frac{2}{12} = -\frac{1}{12} = \zeta(-1) \quad \checkmark \tag{21}$$

4 Conclusion

Through analytic continuation and regularization techniques, we have assigned finite values to divergent series:

- All positive integers: $-\frac{1}{12}$
- Even positive integers: $-\frac{1}{6}$
- Odd positive integers: $\frac{1}{12}$

These results, while counterintuitive from a classical summation perspective, have important applications in quantum field theory, string theory, and other areas of theoretical physics.